Differentiation

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Derivative of functions in the form ax^n

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$$f(x) = ax^n \longrightarrow f'(x) = anx^{n-1} \quad \text{or} \quad \frac{dy}{dx} = anx^{n-1}$$

For example:

$$f(x) = ax^n \longrightarrow f'(x) = anx^{n-1}$$

If $f(x) = 2x^5 \longrightarrow f'(x) = (2)(5)\,x^{5-1} = 10x^4$

If $f(x) = 3x \longrightarrow f'(x) = (3)(1)\,x^{1-1} = 3x^0 = 3$

For any term x except 0, $x^0 = 1$

If $f(x) = 5x^2 \longrightarrow f'(x) = (5)(2)\,x^{2-1} = 10x^1 = 10x$

If $f(x) = 4x^3 \longrightarrow f'(x) = (4)(3)\,x^{3-1} = 12x^2$

Derivative of functions in the form ax^n

The derivative of a constant is zero.

$$f(x) = a \longrightarrow f'(x) = 0$$

For example:

If $f(x) = 2 \longrightarrow f'(x) = 0$

If $f(x) = \frac{1}{2} \longrightarrow f'(x) = 0$

Sum Rule

The sum rule for derivatives states that the derivative of a sum is equal to the sum of the derivatives.

$$f(x) = g(x) + h(x) \longrightarrow f'(x) = g'(x) + h'(x)$$

For example:

If $f(x) = 3x^2 - x + \frac{1}{2}$ find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$f'(x) = 3 \times 2 \times x^{2-1} - 1 \times 1 \times x^{1-1} + 0$$

$$f'(x) = 6x - 1 + 0$$

$$= 6x - 1$$

Second derivative

The second derivative (f'' *or* $\frac{d^2y}{dx^2}$), is the derivative of the derivative (f' **or** $\frac{dy}{dx}$).

To find the second derivative:

1. Find the $f'(x)$.
2. Find derivative of the $f'(x)$.

For example:

If $f(x) = 4x^3 - x + \frac{1}{2}$ find $\frac{d^2y}{dx^2}$.

Step 1 $\frac{dy}{dx} = f'(x) = 4 \times 3 \times x^{3-1} - 1 + 0 = 12x^2 - 1$

Step 2 $\frac{d^2y}{dx^2} = f''(x) = 12 \times 2 \times x^{2-1} - 0 = 24x$

Gradient at a point on a curve by derivative

The gradient at a point on a curve is defined as the gradient of the tangent to the curve at that point.

The derivative function $f'(x)$ gives the gradient (slope) of the tangent to the curve $f(x)$ at any point x.

To find the gradient of a curve at a point on the curve:

1. Find the $f'(x)$.
2. Substitute the x in the $f'(x)$ with the x-coordinate of the point.

For example: Calculate the gradient of the curve $y = x^3 + 9x + 1$ at $x = -2$.

Step 1

$$f'(x) = 1 \times 3 \times x^{3-1} + 9 \times 1 \times x^{1-1} + 0$$
$$= 3x^2 + 9$$

Step 2

The gradient of the curve at $x = -2$ is $f'(-2)$

$$f'(-2) = 3(-2)^2 + 9$$
$$= 12 + 9 = 21$$

Coordinates of the stationary points

To find the coordinate of the stationary points:

1. Find the $f'(x)$.
2. Solve the equation $f'(x) = 0$
3. Substitute x in the function equation by the values of the found x-coordinate

For example: A curve has equation $y = 2x^3 - 3x^2 - 5$.

Find the coordinates of the two stationary points on the curve.

Step 1 $f'(x) = 6x^2 - 6x$

Step 2 $f'(x) = 0 \implies 6x^2 - 6x = 0$

$6x(x-1) = 0$

- $6x = 0 \quad x = 0$
- $x - 1 = 0 \quad x = 1$

Step 3

$y = 2x^3 - 3x^2 - 5 \rightarrow y = 2(0)^3 - 3(0)^2 - 5 \quad y = -5$

$y = 2x^3 - 3x^2 - 5 \rightarrow y = 2(1)^3 - 3(1)^2 - 5 \quad y = -6$

The coordinates of the two stationary points are $(0, -5)$ and $(1, -6)$.

Type of stationary point (turning point)

A **turning point** is a point at which the derivative changes sign.(a curve **changes** from **moving upwards** to **moving downwards**, or vice versa)

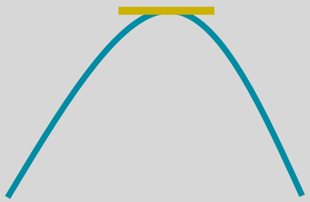
A turning point may be either a relative maximum or a relative minimum.(also known as local minimum and maximum).

If the function is differentiable, then a turning point is a stationary point; however, not all stationary points are turning points.

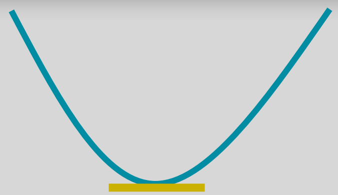
To find the type of stationary point (local maximum - local minimum):

1. Find the coordinates of the stationary points.
2. Substitute the x-coordinate of the stationary points in the second derivative.
   I. If $f''(x) < 0$, it is a local maximum. (**turning point**)
   II. If $f''(x) > 0$, it is a local minimum. (**turning point**)
   III. If $f''(x) = 0$, it is not a local minimum or maximum. (inflection)

Turning point
Local maximum
stationary point

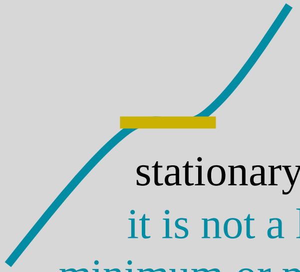


$f''(x) < 0$



stationary point
Local minimum
Turning point

$f''(x) > 0$



stationary point
it is not a local minimum or minimum

$f''(x) = 0$

Example

Find the coordinates of the turning points of the graph $y = x^3 + 3x^2 - 6$.

To find the x-coordinates of the stationary points:

Step 1 $f'(x) = 3x^2 + 6x$

Step 2 $f'(x) = 0$ $3x^2 + 6x = 0$ $3x(x+2) = 0$

$3x = 0$ $\boldsymbol{x = 0}$

$x + 2 = 0$ $\boldsymbol{x = -2}$

stationary points

To find the type of stationary point (local maximum - local minimum):

Step 1 $f''(x) = 6x + 6$

Step 2 $f''(0) = 6(0) + 6 = 6 \quad > 0$ Local minimum $(0, -6)$

$f(0) = (0)^3 + 3(0)^2 - 6 = -6$

$f''(-2) = 6(-2) + 6 = -6 \quad < 0$ Local maximum $(-2, -2)$

$f(-2) = (-2)^3 + 3(-2)^2 - 6 = -2$