Question 1

A curve has equation $y = x^3 - 3x + 4$.

a) Find the coordinates of its two stationary points.
b) Determine whether each stationary point is a maximum or a minimum.

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a) $f'(x) = 3x^2 - 3$

1 Find the $f'(x)$.

$3x^2 - 3 = 0$ $\quad x^2 = 1$ $\quad x = \pm 1$

2 Solve the equation $f'(x) = 0$

$f(1) = (1)^3 - 3(1) + 4$ $\quad y = 2$

3 Substitute x in the function equation by the values of the found x-coordinate.

$f(-1) = (-1)^3 - 3(-1) + 4$ $\quad y = 6$

The coordinates of its two stationary points are: $(1,2), (-1,6)$

b) $(1,2), (-1,6)$

1 Find the coordinates of the stationary points.

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

2 Substitute the x-coordinate of the stationary points in the second derivative.

I. If $f''(x) < 0$, it is a local maximum.
II. If $f''(x) > 0$, it is a local minimum.
III. If $f''(x) = 0$, it is not a local minimum or minimum. (inflection)

$f''(1) = 6(1) = 6 > 0$ Minimum

$f''(-1) = 6(-1) = -6 < 0$ Maximum

The coordinates of its two return points are: $(1,2)$ Minimum, $(-1,6)$ Maximum

Question 2

Find the minimum value of $y = 5x^2 - 20x + 15$.

Differentiate: $\frac{dy}{dx} = 10x - 20$.

Set $\frac{dy}{dx} = 0$: $10x - 20 = 0$ gives $x = 2$.

$y(2) = 5(2)^2 - 20(2) + 15 = 20 - 40 + 15 = -5$.

Minimum value: -5.

Question 3

Find the x-coordinate of the turning point of $y = 2x^2 - 8x + 5$.

Differentiate: $\frac{dy}{dx} = 4x - 8$.

Set $\frac{dy}{dx} = 0$: $4x - 8 = 0$ gives $x = 2$.

x-coordinate = 2.

Question 4

A curve has equation $y = x^3 - x^2 - x$.

a) Find the coordinates of its two stationary points.
b) Determine whether each stationary point is a maximum or a minimum.

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a) $f'(x) = 3x^2 - 2x - 1$

$3x^2 - 2x - 1 = 0$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = 1$ $x = \frac{-1}{3}$

$f(1) = (1)^3 - (1)^2 - (1)$ $y = -1$

$f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 - \left(\frac{-1}{3}\right)^2 - \left(-\frac{1}{3}\right)$ $y = 0.19$

The coordinates of its two stationary points are: $(1, -1), (\frac{-1}{3}, 0.19)$

1. Find the $f'(x)$.
2. Solve the equation $f'(x) = 0$
3. Substitute x in the function equation by the values of the found x-coordinate.

b) $(1, -1), (\frac{-1}{3}, 0.19)$

$f'(x) = 3x^2 - 2x - 1$

$f''(x) = 6x - 2$

$f''(1) = 6(1) = 6 - 2 = 4 > 0$ Minimum

$f''(-1) = 6(-1) = -6 - 2 = -8 < 0$ Maximum

1. Find the coordinates of the stationary points.
2. Substitute the x-coordinate of the stationary points in the second derivative.
   I. If $f''(x) < 0$, it is a local maximum.
   II. If $f''(x) > 0$, it is a local minimum.
   III. If $f''(x) = 0$, it is not a local minimum or minimum. (inflection)

The coordinates of its two return points are: $(1, -1)$ Minimum, $(-1, 0.19)$ Maximum

Question 5

A curve has equation $y = x^3 - 3x^2$.

a) Find the coordinates of its two turning points.
b) Determine whether each stationary point is a maximum or a minimum.

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a) $f'(x) = 3x^2 - 6x$

$3x^2 - 6x = 0$ $\quad 3x(x-2) = 0$ $\quad x = 0$, $x = 2$

$f(0) = (0)^3 - 3(0)^2$ $\quad y = 0$

$f(2) = (2)^3 - 3(2)^2$ $\quad y = -4$

The coordinates of its two stationary points are: $(0,0), (2,-4)$

1. Find the $f'(x)$.
2. Solve the equation $f'(x) = 0$
3. Substitute x in the function equation by the values of the found x-coordinate.

b) $(0,0), (2,-4)$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6 = 0 - 6 = -4 < 0$ Maximum

$f''(2) = 6(2) - 6 = 12 - 6 = 6 > 0$ Minimum

1. Find the coordinates of the stationary points.
2. Substitute the x-coordinate of the stationary points in the second derivative.
   I. If $f''(x) < 0$, it is a local maximum..
   II. If $f''(x) > 0$, it is a local minimum.
   III. If $f''(x) = 0$, it is not a local minimum or minimum. (inflection)

The coordinates of its two return points are: $(2,-4)$Minimum, $(0,0)$Maximum

Question 6

A curve has equation $y = \frac{1}{4}x^4 + \frac{2}{3}x^3 - 1$.

a) Find the coordinates of its stationary points.
b) Determine whether each stationary point is a maximum or a minimum or neither.

a) $f'(x) = x^3 - 2x^2$

$x^3 - 2x^2 = 0$ $\quad x^2(x+2) = 0$ $\quad x = 0$, $x = -2$

$f(0) = \frac{1}{4}(0)^4 + \frac{2}{3}(0)^3 - 1$ $\quad y = -1$

$f(-2) = \frac{1}{4}(-2)^4 + \frac{2}{3}(-2)^3 - 1$ $\quad y = -2.33$

The coordinates of its two stationary points are: $(0, -1), (-2, -2.33)$

1. Find the $f'(x)$.
2. Solve the equation $f'(x) = 0$
3. Substitute x in the function equation by the values of the found x-coordinate.

b) $(0, -1), (-2, -2.33)$

$f'(x) = x^3 - 2x^2$

$f''(x) = 3x^2 - 4x$

$f''(0) = 3(0)^2 - 4(0) = 0$ it is not a local minimum or minimum.

$f''(-2) = 3(-2)^2 - 4(-2) = 20 > 0$ Minimum

The coordinates of its two return points are: $(-2, -2.33)$ Minimum

1. Find the coordinates of the stationary points.
2. Substitute the x-coordinate of the stationary points in the second derivative.
   I. If $f''(x) < 0$, it is a local maximum..
   II. If $f''(x) > 0$, it is a local minimum.
   III. If $f''(x) = 0$, it is not a local minimum or minimum. (inflection)

Question 7

A curve has equation $y = x(x-2)(x+1)$.

a) Find the coordinates of its two stationary points.
b) Determine whether each stationary point is a maximum or a minimum.

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a) $y = x(x-2)(x+1) = (x^2 - 2x)(x+1)$

$f(x) = x^3 - x^2 - 2x$

$f'(x) = 3x^2 - 2x - 2$

$3x^2 - 2x - 2 = 0$ $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad x = -0.55$, $x = 1.22$

$f(-0.55) = (-0.55)^3 - (-0.55)^2 - 2(-0.55)$ $\quad y = 0.63$

$f(1.22) = (1.22)^3 - (1.22)^2 - 2(1.22)$ $\quad y = -2.11$

The coordinates of its two stationary points are: $(1.22, -2.11), (-0.55, 0.63)$

1. Find the $f'(x)$.
2. Solve the equation $f'(x) = 0$
3. Substitute x in the function equation by the values of the found x-coordinate.

b) $(1.22, -2.11), (-0.55, 0.63)$

$f'(x) = 3x^2 - 2x - 2$

$f''(x) = 6x - 2$

$f''(1.22) = 6(1.22) - 2 > 0$ Minimum

$f''(-0.55) = 6(-0.55) - 2 < 0$ Maximum

The coordinates of its two return points are: $(1.22, -2.11)$Minimum, $(-0.55, 0.63)$Maximum

1. Find the coordinates of the stationary points.
2. Substitute the x-coordinate of the stationary points in the second derivative.
   I. If $f''(x) < 0$, it is a local maximum..
   II. If $f''(x) > 0$, it is a local minimum.
   III. If $f''(x) = 0$, it is not a local minimum or minimum. (inflection)

Question 8

A curve has equation $y = \frac{1}{5}x^5 + x^4 - 2$.

a) Find the x-coordinates of its turning points.
b) Determine whether each stationary point is a maximum or a minimum.

$f'(x) = x^4 + 4x^3$

$f'(x) = 0 \quad x^4 + 4x^3 = 0$

$x^3(x+4) = 0$

→ $x^3 = 0 \quad x = 0$

→ $x + 4 = 0 \quad x = -4$

1 Find the $f'(x)$.

2 Solve the equation $f'(x) = 0$

3 Substitute x in the function equation by the values of the found x-coordinate.

in the question, x coordinates of the turning points are only asked.

$f''(x) = 4x^3 + 12x^2$

$x = 0 \quad f''(0) = 4(0)^3 + 12(0)^2 = 0$

It is not a local minimum or maximum

$x = -4 \quad f''(x) = 4(-4)^3 + 12(-4)^2 = -65 < 0$

Local maximum

The x-coordinate of the turning points on the graph of y is -4.

1 Find the coordinates of the stationary points.

2 Substitute the x-coordinate of the stationary points in the second derivative.

I. If $f''(x) < 0$, it is a local maximum..

II. If $f''(x) > 0$, it is a local minimum.

III. If $f''(x) = 0$, it is not a local minimum or minimum. (inflection)

Question 9

A curve has equation $y = x^3 - 3x^2 + 3x$.

Find the coordinates of its turning points.

$f'(x) = 3x^2 - 6x + 3$

$f'(x) = 0 \qquad 3x^2 - 6x + 3 = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(3)}}{2(3)}$

$x = 1$

$x = 1$

Repeated roots

in the question, x coordinates of the turning points are only asked.

$f''(x) = 6x - 6$

$x = 1 \qquad f''(0) = 6(0) - 6 = -6 \qquad = 0$

It is not a local minimum or maximum

This graph does not have a turning point.

1 Find the $f'(x)$.

2 Solve the equation $f'(x) = 0$

3 Substitute x in the function equation by the values of the found x-coordinate.

1 Find the coordinates of the stationary points.

2 Substitute the x-coordinate of the stationary points in the second derivative.

I. If $f''(x) < 0$, it is a local maximum..

II. If $f''(x) > 0$, it is a local minimum.

III. If $f''(x) = 0$, it is not a local minimum or minimum. (inflection)

Question 10

A curve has equation $y = x^3 + ax + b$.
The coordinates of the stationary points are $(-1, -k)$, $(1, k+2)$.

Calculate the values of a, b and k.

We have 3 unknown variables a, b and k, so we need to make 3 equations and then solve simultaneous equations.

First equation:

> The x-coordinates of the stationary points are roots of the equation $f'(x) = 0$.

The coordinates of the stationary points are $(-1, -k)$, $(1, k+2)$

So $x = 1$ and $x = -1$ are roots of the equation $f'(x) = 0$

$f'(x) = 3x^2 + a$ $\quad \begin{cases} 3x^2 + a = 0 \\ x = 1 \end{cases} \longrightarrow a = \mathbf{-3}$

Note that by substitute x to -1 also will be a=3.

Second and third equations:

> Stationary points lie on the curve, so the coordinates of Stationary points satisfy the equation y $= x^3 + ax$ +b.

$(-1, -k) \longrightarrow y = x^3 + ax + b \qquad -k = (-1)^3 + a(-1) + b$

$(1, k+2) \longrightarrow y = x^3 + ax + b \qquad k + 2 = (1)^3 + a(1) + b$

$\begin{cases} -k = -1 - a + b \\ k + 2 = 1 + a + b \\ a = -3 \end{cases} \longrightarrow \begin{cases} -k = -1 - (-3) + b \\ k + 2 = 1 + (-3) + b \end{cases} \quad \begin{cases} -k = 2 + b \\ k + 2 = -2 + b \end{cases} \qquad -k - 2 = b$

Substitute b to $-k - 2$

$\longrightarrow k + 2 = -2 + (-k - 2) \qquad k = \mathbf{-3}$

$\longrightarrow -(-3) - 2 = b$

$b = \mathbf{1}$