Functions

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Functions

A function is a mathematical concept that defines a relationship between two variables, such that for every input value of one variable, there is a unique output value of the other variable. In other words, a function takes an input value, performs a set of operations on it, and produces a corresponding output value.

The standard way of representing a function is using the notation *f(x)*, where '*f*' is the name of the function and '*x*' is the input variable.

For example, the function $f(x) = x + 3$ represents a relationship between the input variable *x* and the output variable y, where y is obtained by adding 3 to *x*. So, if we input the value $x = 2$ into the function, we get $f(2) = 2 + 3 = 5$ as the corresponding output value.

f(x) is the output value(also called the dependent variable)

f is the name of the function

$$f(x) = x + 3$$

Formula or equation of function *f*

x is the input value
(also called the independent variable)

Function values

Function values, also known as output values or dependent variable values, are the values obtained when an input value is plugged into a function. In other words, given a function f(x), the function value at a specific input value x is the output value obtained by evaluating the function at x.

Finding function values is the process of evaluating the function for given values of x.

Note To find the function value substitute the variable of the function with a number or expression given.

For example:

if $f(x) = 3x - 4$, calculate $f(-5)$ and $f(2x+1)$

$f(-5) =$

$f(x) = 3x - 4$

$f(-5) = 3(-5) - 4$

$f(-5) = -19$

$f(2x+1) =$

$f(x) = 3x - 4$

$f(2x+1) = 3(2x+1) - 4$

$f(2x+1) = 6x + 3 - 4$

$f(2x+1) = 6x - 1$

Composite functions

A composite function is a function that is formed by combining two or more functions in a specific way.

(A composite function is a function that is written inside another function).

$$\mathrm{fg}(x) = f\big(g(x)\big)$$

For example:

If $f(x) = 3x - 5$ and $g(x) = x - 1$ find $fg(x)$.

$$fg(x) = f(g(x))$$

$$= f(x - 1)$$

$$= 3(x - 1) - 5$$

$$= 3x - 3 - 5$$

$$= 3x - 8$$

Example

$f(x) = 3^{x-2} - 4$

Find $ff(4)$.

$$ff(4) = f\big(f(4)\big)$$

$$f(4) = 3^{4-2} - 4 = 5$$

$$= f(5)$$

$$= 3^{5-2} - 4$$

$$= 3^3 - 4$$

$$= 27 - 4$$

$$= 23$$

$$ff(4) = 23$$

Example

$f(x) = x + 5$ $g(x) = x^2 - 4$

Solve $g(x) + x = f(x + 7)$.

$$g(x) + x = f(x + 7)$$

$$(x^2 - 4) + x = (x + 7) + 5$$

$$x^2 - 4 + \cancel{x} - \cancel{x} - 12 = 0$$

$$x^2 - 16 = 0$$

$$x^2 = 16 \qquad x = \pm 4$$

Inverse functions

An inverse function is a function that undoes the effect of another function. In other words, if the original function transforms an element **a** into an element **b**, the inverse function transforms the element **b** back into the element **a**, returning it to its original state. The inverse of a function $f(x)$ is symbolized as $f^{-1}(x)$, with the superscript "-1".

$$\text{If } f(x) = y, \quad \text{then} \quad f^{-1}(y) = x$$

To find the inverse of a function:

1. Replace the function notation with "y" (or another variable of your choice).
2. Swap the variables, replacing every "x" with a "y" and every "y" with an "x".
3. Solve the equation for "y".
4. Replace y with $f^{-1}(x)$ to get the inverse function.

The inverse of an inverse function.

$$(f^{-1})^{-1} = f$$

Example

If $f(x) = 3x - 5$, calculate $f^{-1}(x)$ and $f^{-1}(5)$.

1. Replace the function notation with "y" (or another variable of your choice).

$$y = 3x - 5$$

2. Swap the variables, replacing every "x" with a "y" and every "y" with an "x".

$$x = 3y - 5$$

3. Solve the equation for "y".

$$\frac{x+5}{3} = y$$

4. replace y with $f^{-1}(x)$.

$$\frac{x+5}{3} = f^{-1}(x)$$

$\boldsymbol{f^{-1}(5)}$

$$f^{-1}(5) = \frac{5+5}{3}$$

$$f^{-1}(5) = \frac{10}{3}$$

Example

$f(x) = x^3 + 3$

Find $f^{-1}(x)$ and $f^{-1}(2)$

$f(x) = x^3 + 3$

$y = x^3 + 3$

$x = y^3 + 3$

$x - 3 = y^3$

$\sqrt[3]{x-3} = y$

$\sqrt[3]{x-3} = f^{-1}(x)$

$f^{-1}(2)$

$f^{-1}(x) = \sqrt[3]{x-3}$

$f^{-1}(2) = \sqrt[3]{2-3}$

$= \sqrt[3]{-1}$

$= -1$

$\mathrm{f}^{-1}(2) = -1$

Example

$$f(x) = \frac{2x - 3}{x + 1}$$

Find $f^{-1}(x)$ and $f^{-1}(1)$

$$f(x) = \frac{2x - 3}{x + 1}$$

$$y = \frac{2x - 3}{x + 1}$$

$$x = \frac{2y - 3}{y + 1}$$

$$x \times (y + 1) = \frac{2y - 3}{y + 1} \times (y + 1)$$

$$xy + x = 2y - 3$$

$$xy - 2y = -x - 3$$

$$y(x - 2) = -x - 3$$

$$y = \frac{-x - 3}{x - 2}$$

$$f^{-1}(x) = \frac{-x - 3}{x - 2}$$

$$f^{-1}(1) = \frac{-(1) - 3}{1 - 2}$$

$$f^{-1}(1) = \frac{-4}{-1} = 4$$

Example

$f(x) = \sqrt[3]{2x+1}$

Find $f^{-1}(x)$ and $f^{-1}(-1)$

$$f(x) = \sqrt[3]{2x+1}$$

$$y = \sqrt[3]{2x+1}$$

$$x = \sqrt[3]{2y+1}$$

$$x^3 = 2y+1$$

$$x^3 - 1 = 2y$$

$$\frac{x^3-1}{2} = y$$

$$\frac{x^3-1}{2} = f^{-1}(x)$$

$$f^{-1}(-1)$$

$$f^{-1}(x) = \frac{x^3-1}{2}$$

$$f^{-1}(-1) = \frac{(-1)^3-1}{2}$$

$$f^{-1}(-1) = -1$$

Example

$f(x) = 2x + 5$

Find $f^{-1}f(x+3)$.

$$f(x) = 2x + 5$$

$$y = 2x + 5$$

$$x = 2y + 5$$

$$x - 5 = 2y$$

$$\frac{x-5}{2} = y$$

$$\frac{x-5}{2} = f^{-1}(x)$$

$$f^{-1}f(x+3) = f^{-1}(f(x+3))$$

$$f(x) = 2x + 5 \longrightarrow \mathrm{f}(x+3) = 2(x+3) + 5$$

$$= 2x + 11$$

$$f^{-1}f(x+3) = f^{-1}(2x+11)$$

$$f^{-1}(f(x+3)) = \frac{(2x+11) - 5}{2}$$

$$= \frac{2x+6}{2} = \frac{2x}{2} + \frac{6}{2}$$

$$= x + 3$$

$$\mathrm{f}^{-1}\mathrm{f}(\mathrm{x}+3) = \mathrm{x} + 3$$

Example

$f(x) = 2^x$

Find x when $f^{-1}(x) = -2$.

If $f(x) = y$, then $f^{-1}(y) = x$

Since $f^{-1}(x) = -2$ by the definition of inverse functions, $f(-2) = x$.

This means that when $x = -2$, the output of the function $f(x)$ should equal x.

Using the given function $f(x) = 2^x$

$$f(-2) = x$$

$$2^{-2} = x$$

$$\frac{1}{2^2} = x$$

$$\frac{1}{4} = x$$

Inverse functions
Graphical Interpretation

The graph of a function and its inverse are reflections of each other in the line $y=x$

To sketch the graph of an inverse function:

1. Construct a table for the origin function then plot points of the table on the coordinate plane.
2. Draw the Line $y = x$.
3. The inverse function graph reflects the original function's graph across the line $y = x$.
   To reflect a point (a, b) on the graph of $f(x)$, plot the point (b, a) on the graph of $f^{-1}(x)$.