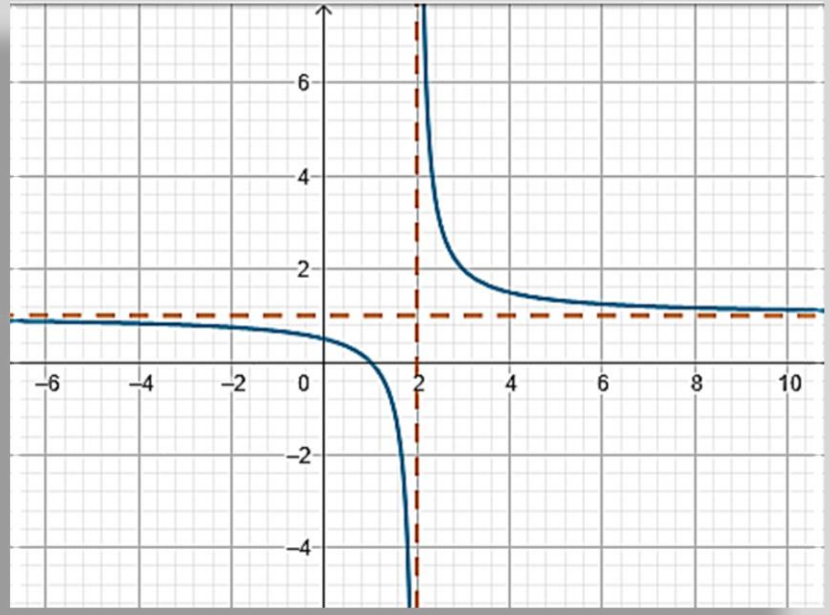
Asymptote

An **asymptote** is a straight line that a curve approaches but never touches or crosses as the curve extends to infinity. Asymptotes can be:

1) **Horizontal asymptotes:** The curve approaches a fixed horizontal line as $x \to \pm\infty$.

2) **Vertical asymptotes:** The curve approaches a vertical line as x approaches a specific value where the function is undefined.

$x = 2$ vertical asymptote



$y = 1$ horizontal asymptote

Sketch reciprocal function

$$y = \frac{a}{x} + c \quad \textbf{or} \qquad f(x) = \frac{1}{x} + c$$

This function is called **the reciprocal function.**

The graph of a reciprocal function **is a hyperbola.**

The reciprocal functions have a vertical asymptote and a horizontal asymptote.

$x = 0$ The equation of vertical asymptote

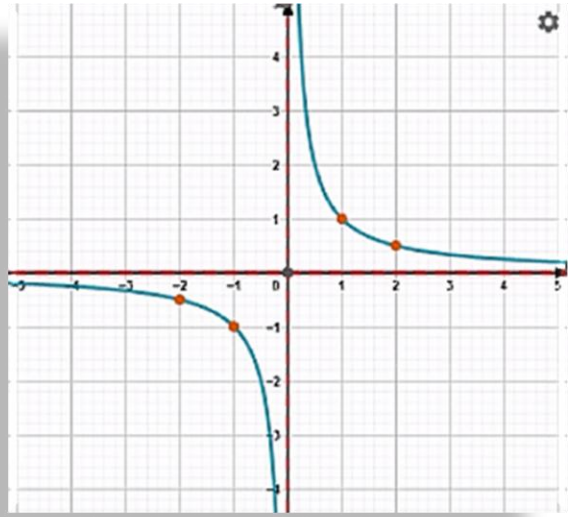
$y = 0$ The equation of horizontal asymptote

To sketch the reciprocal function $y = \frac{1}{x}$ you always can use this table:

x	-2	-1	0	1	2
y	-0.5	-1	vertical asymptote	1	0.5

$y = \frac{1}{x}$ $\qquad$ $y = \frac{1}{x}$

$y = \frac{1}{-2} = -0.5$ $\qquad$ $y = \frac{1}{-1} = -1$



EXAMPLE

Sketch the graph of the function $y = \frac{-2}{x}$.

This function is called **the reciprocal function.**

The graph of a reciprocal function **is a hyperbola.**

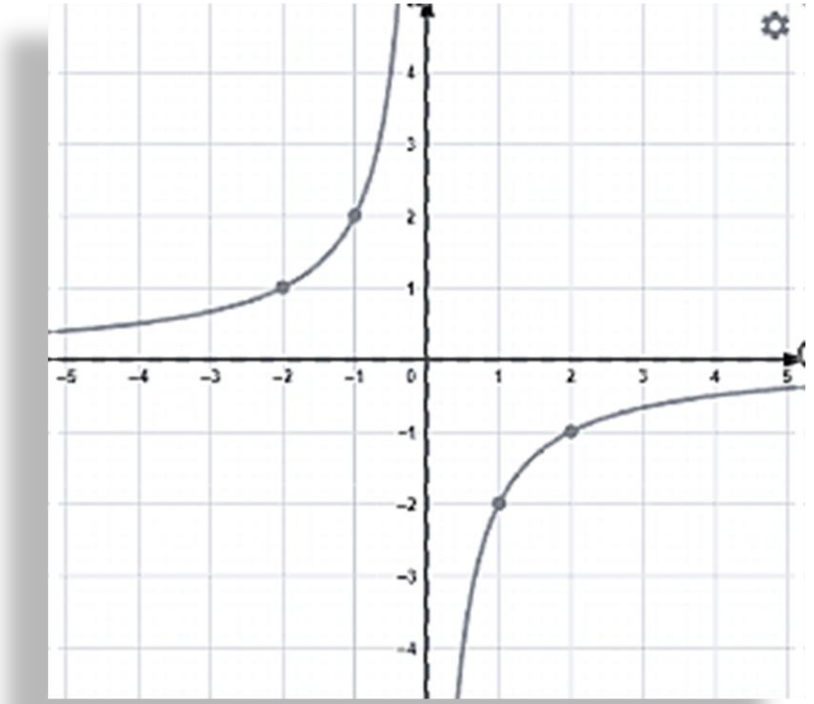
$x = 0$ The equation of vertical asymptote

$y = 0$ The equation of horizontal asymptote

x	-2	-1	0	1	2
y	1	2	asymptote	-2	-1

$y = \frac{-2}{x}$ $y = \frac{-2}{x}$ $y = \frac{-2}{x}$ $y = \frac{-2}{x}$ $y = \frac{-2}{x}$

$y = \frac{-2}{-2} = 1$ $y = \frac{-2}{-1} = 2$ $y = \frac{-2}{0}$ asymptote $y = \frac{-2}{1} = -2$ $y = \frac{-2}{2} = -1$



Sketch exponential function

$$f(x) = ab^x + c \qquad \textbf{or} \qquad y = ab^x + c$$

This function is called **the exponential function.**

The exponential functions have only a horizontal asymptote. $y = c$

For example, if $a = 1, b = 2$ *and* $c = 0$, we construct tables of values and draw the graph $y = 2^x$.

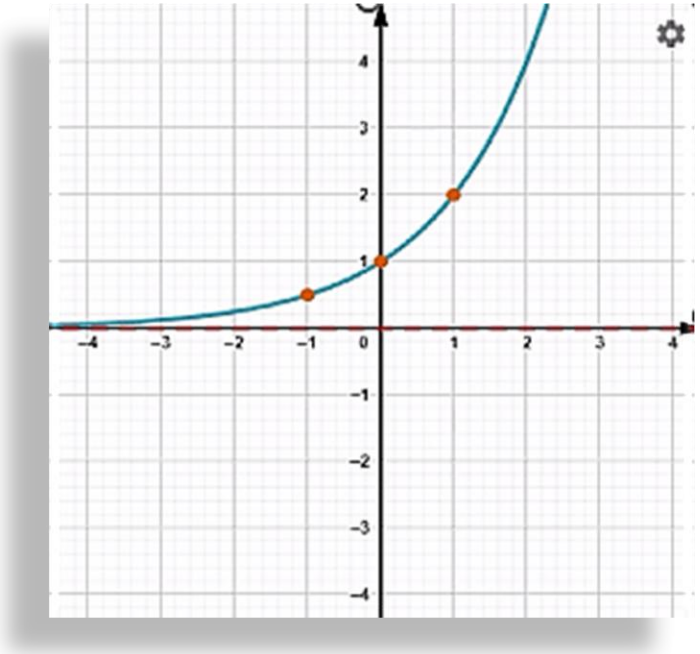
x	-1	0	1
y	0.5	1	2

$y = 2^x$ $\quad y = 2^x$ $\quad y = 2^x$

$y = 2^{-1}$ $\quad y = 2^0$ $\quad y = 2^1$

$y = \frac{1}{2} = 0.5$ $\quad y = 1$ $\quad y = 2$

The equation of horizontal asymptote $y = 0$



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EXAMPLE

sketch the graph of the function $\boldsymbol{y = 3^{-x}}$.

This function is an **exponential function.**

x	-1	0	1
y	3	1	0.33

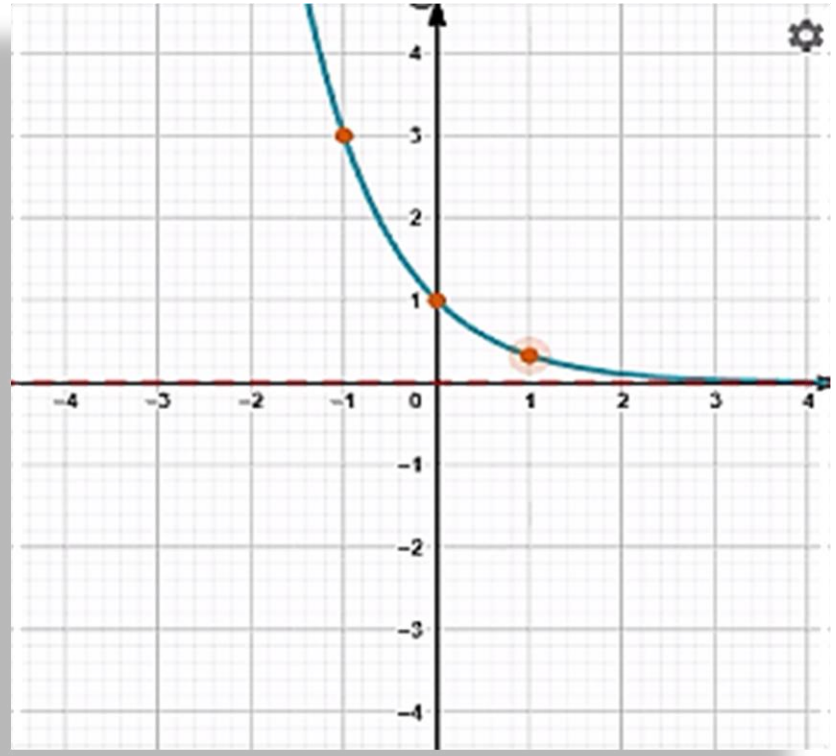
$y = 3^{-x}$
$y = 3^{-(-1)}$
$y = 3$

$y = 3^{-x}$
$y = 3^{0}$
$y = 1$

$y = 3^{-x}$
$y = 3^{-(1)}$
$y = \frac{1}{3} = 0.33$

The equation of horizontal asymptote

$y = 0$



EXAMPLE

sketch the graph of the function $\boldsymbol{y = 2 \times 3^x + 1}$.

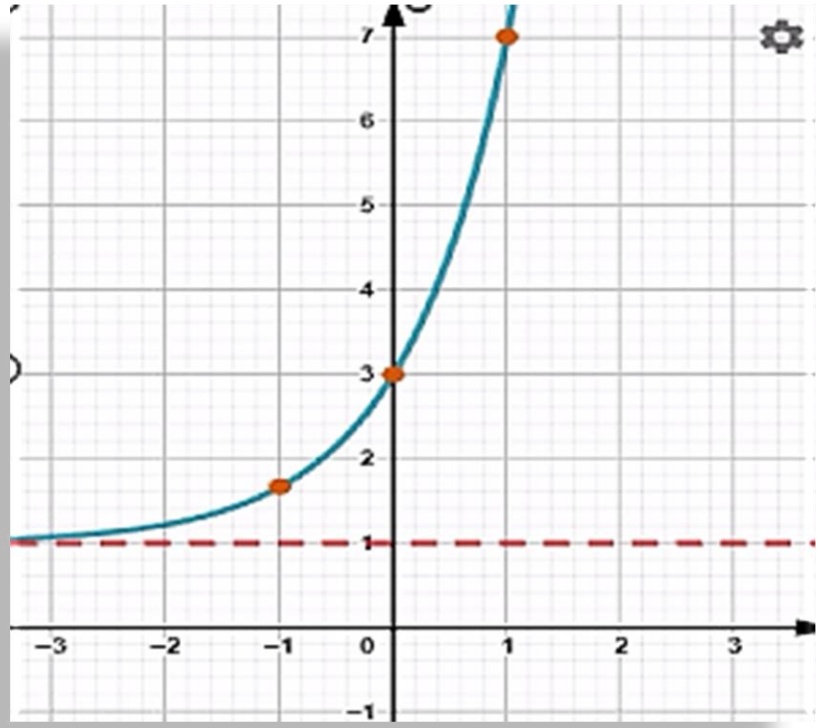
x	-1	0	1
y	1.67	3	7

$y = 2 \times 3^x + 1$ $y = 2 \times 3^x + 1$ $y = 2 \times 3^x + 1$

$y = 2 \times 3^{(-1)} + 1$ $y = 2 \times 3^{(0)} + 1$ $y = 2 \times 3^{(1)} + 1$

$y = \frac{2}{3} + 1 = 1 \cdot 67$ $y = 3$ $y = 7$

The equation of horizontal asymptote $y = 1$



Example

Sketch the curve of $y = -x^3 + 4x^2 - 4x$.

Step 1

Find the intersection of the graph with the coordinate axes.

- To find the x-axis intercept of a function, substitute y = 0 and solve for x.
- To find the y-axis intercept of a function, substitute x = 0 and solve for y.

$y = 0$

$y = -x^3 + 4x^2 - 4x$

$0 = -x^3 + 4x^2 - 4x$

$x(-x^2 + 4x - 4) = 0$

$x = 0$

$(-x^2 + 4x - 4) = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = 2$

$x = 2$

The repeated root

$x = 0$

$y = -x^3 + 4x^2 - 4x$

$y = (0)^3 + 4(0)^2 - 4(0)$

$y = 0$

The coordinates of the points where the graph intersects the x-axis. $\begin{cases} x = 2 \\ y = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \end{cases}$

Note:
The repeated root $x = 2$ implies that the graph just touches the x-axis at this point.

The coordinates of the point where the graph intersects the y-axis. $\begin{cases} x = 0 \\ y = 0 \end{cases}$

The coordinates of the points where the graph intersects the x-axis. $\begin{cases} x = 2 \\ y = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \end{cases}$

The coordinates of the point where the graph intersects the y-axis. $\begin{cases} x = 0 \\ y = 0 \end{cases}$

Step 2 Find the stationary points

- Find the $f'(x)$.
- Solve the equation $f'(x) = 0$
- Substitute x in the function equation by the values of the found x-coordinate

$$y' = -3x^2 + 8x - 4$$

$$0 = -3x^2 + 8x - 4 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-8 \pm \sqrt{(8)^2 - 4(-3)(-4)}}{2(-3)} \qquad \begin{matrix} x = 2 \\ x = \frac{2}{3} \end{matrix}$$

$$x = 2 \qquad x = \frac{2}{3}$$

$$y = -(2)^3 + 4(2)^2 - 4(2) = 0 \qquad y = -(\frac{2}{3})^3 + 4(\frac{2}{3})^2 - 4\left(\frac{2}{3}\right) = -1.18$$

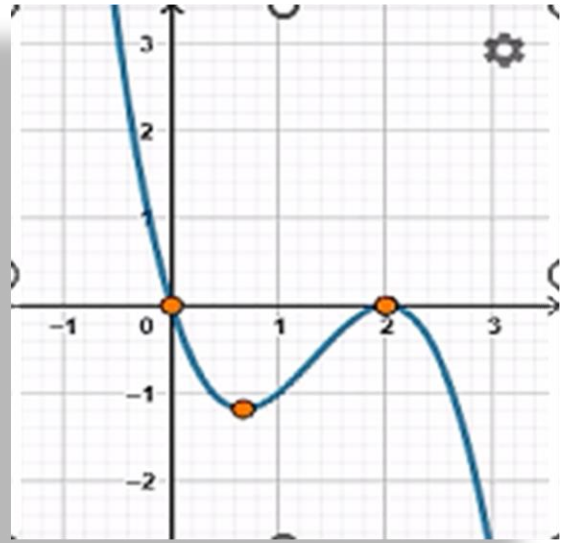
The coordinates of the points where the graph intersects the x-axis. $\begin{cases} x = 2 \\ y = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \end{cases}$

The coordinates of the point where the graph intersects the y-axis. $\begin{cases} x = 0 \\ y = 0 \end{cases}$

stationary points $x = 2 \quad y = 0$ $x = \frac{2}{3} \quad y = -1.18$

Step 3 Graph's shape

- If $a > 0$ the graph's shape is  for two or three x-axis or  for one x-axis.
- If $a < 0$ the graph's shape is  for two or three x-axis or  for one x-axis.



Example

Sketch the curve of $y = x^3 + 8$.

$y = x^3 + 8$

$\boldsymbol{y = 0}$

$0 = x^3 + 8 \qquad x^3 = -8 \qquad x = \sqrt[3]{-8} \qquad x = -2$

The coordinates of the points where the graph intersects the x-axis. $\begin{cases} x = -2 \\ y = 0 \end{cases}$

$\boldsymbol{x = 0}$

$y = (0)^3 + 8 \qquad y = 8$

The coordinates of the point where the graph intersects the y-axis. $\begin{cases} x = 0 \\ y = 8 \end{cases}$

$y' = 3x^2$

$0 = 3x^2 \qquad x = 0$

$y = (0)^3 + 8$ stationary point $x = 0 \quad y = 8$

If $a > 0$ the graph's shape is  for two or three x-axis or  for one x-axis.