Factorise

Factorisation is the process of writing an expression as a product of its factors.
Factorisation is the reverse process of expansion.

Expansion →

$$3x(5 - 2x + x^2) = 15x - 6x^2 + 3x^3$$

← Factorisation

In expansions we must remove brackets, whereas in factorisation we must insert brackets.

There are different methods of factorising algebraic expressions.

The most common ones are:

- Factorising with common factors.
- Factorising by Grouping.
- Difference of two squares factorisation.
- Trinomials in Quadratic form $(ax^2 + bx + c)$.

Common Factor Factorisation

Common factor factorisation is when you take out the highest common factor from each term in the expression.

To factorise an algebraic expression involving a few terms we look for the highest common factor HCF (or Greatest Common Factor (GCF)) of the terms and write it in front of a set of brackets. Find the missing numbers in the brackets by dividing each term by the HCF.

At the time of factorizing, the first method you should check is the common factor.

Factorisation by Taking Out the Common Factors

For example:

$$\mathbf{4x + 12} = 4\left(\frac{4x}{4} + \frac{12}{4}\right) = 4(x+3) \qquad HCF\,(Highest\ Common\ Factor)\ is\ 4$$

$$\mathbf{3x^3 + 9x^2 - 12x} = 3x\left(\frac{3x^3}{3x} + \frac{9x^2}{3x} - \frac{12x}{3x}\right) = 3x(x^2 + 3x - 4) \qquad HCF\ is\ 3x$$

$$\mathbf{2(x-1) + 3x(x-1)} = (x-1)\left(\frac{2(x-1)}{(x-1)} + \frac{3x(x-1)}{(x-1)}\right) \qquad HCF\ is\ (x-1)$$

$$= (x-1)(2+3x)$$

Example

Factorise the following expressions.

a) $x^2 + xy$

b) $4x^3 + 6x^2y$

a)

Common factor

$$x^2 + xy = x\left(\frac{x^2}{x} + \frac{xy}{x}\right)$$

$$= x(x + y)$$

b)

Common factor

$$4x^3 + 6x^2y = 2x^2\left(\frac{4x^3}{2x^2} + \frac{6x^2y}{2x^2}\right)$$

$$= 2x^2(2x + 3y)$$

Example

Factorise the following expressions.

a) $xy - x^2$

b) $x^2y^3 - x^3y^2$

a)

Common factor

$$xy - x^2 = x\left(\frac{xy}{x} - \frac{x^2}{x}\right)$$

$$= x(y - x)$$

b)

Common factor

$$x^2y^3 - x^3y^2 = x^2y^2\left(\frac{x^2y^3}{x^2y^2} - \frac{x^3y^2}{x^2y^2}\right)$$

$$= x^2y^2(y - x)$$

Example

Factorise the following expressions.

a) $xyz - x^2y$

b) $x^2y^3z^4 - x^3y^2z$

a)

Common factor

$$xyz - x^2y = xy\left(\frac{xyz}{xy} - \frac{x^2y}{xy}\right)$$

$$= xy(z - x)$$

b)

Common factor

$$x^2y^3z^4 - x^3y^2z = x^2y^2z\left(\frac{x^2y^3z^4}{x^2y^2z} - \frac{x^3y^2z}{x^2y^2z}\right)$$

$$= x^2y^2z(yz^3 - x)$$

Example

Factorise the following expressions.

a) $25xy - 15x^2y + 5x$

b) $9x^3 - 12x^3y^2 + 3x^2$

a)

Common factor

$$25xy - 15x^2y + 5x = 5x\left(\frac{25xy}{5x} - \frac{15x^2y}{5x} + \frac{5x}{5x}\right)$$

$$= 5x(5y - 3xy + 1)$$

b)

Common factor

$$9x^3 - 12x^3y^2 + 3x^2 = 3x^2\left(\frac{9x^3}{3x^2} - \frac{12x^3y^2}{3x^2} + \frac{3x^2}{3x^2}\right)$$

$$= 3x^2(3x - 4xy^2 + 1)$$

Factorising by Grouping

Group factorising is used when an expression has four terms, and terms can be grouped in pairs to reveal common factors. This technique involves grouping terms, factoring out the common factors within each group, and then looking for a common binomial factor.

For example: Factorise the following by grouping:

$$\mathbf{2x + 2y + ax + ay}$$

We can group the first two terms, and the last two terms together as follows:

$$\mathbf{2x + 2y} + \mathbf{ax + ay} =$$

Now we can factor out the common factor from each pair:

$$2\left(\frac{2x}{2} + \frac{2y}{2}\right) + a\left(\frac{ax}{a} + \frac{ay}{a}\right) =$$

$$2(x + y) + a(x + y) =$$

Notice that both terms now have a common factor of $(x + y)$.
We can factor this out to get:

$$(x + y)\left(\frac{2(x + y)}{(x + y)} + \frac{a(x + y)}{(x + y)}\right) = (x + y)(2 + a)$$

Example

Factorise the following by grouping:

$$2x^2 + 8x + ax + 4a$$

$$\mathbf{2x^2 + 8x} + \mathbf{ax + 4a} = 2x\left(\frac{2x^2}{2x} + \frac{8x}{2x}\right) + a\left(\frac{ax}{a} + \frac{4a}{a}\right)$$

$$= 2x(x+4) + a(x+4)$$

$$= (x+4)\left(\frac{2x(x+4)}{(x+4)} + \frac{a(x+4)}{(x+4)}\right)$$

$$= (x+4)(2x+a)$$

Example

Factorise the following by grouping:

$$2x + 8y + 3x^2 + 12xy$$

$$2x + 8y + 3x^2 + 12xy = 2\left(\frac{2x}{2} + \frac{8y}{2}\right) + 3x\left(\frac{3x^2}{3x} + \frac{12xy}{3x}\right)$$

$$= 2(x + 4y) + 3x(x + 4y)$$

$$= (x + 4y)\left(\frac{2(x + 4y)}{(x + 4y)} + \frac{3x(x + 4y)}{(x + 4y)}\right)$$

$$= (x + 4y)(2 + 3x)$$

Example

Factorise the following by grouping:

$$2x - 6xz - 3y + 9yz$$

If the third term has been negative, take the common factor with the negative sign.

$$2x - 6xz - 3y + 9yz = 2x\left(\frac{2x}{2x} - \frac{6xz}{2x}\right) - 3y\left(\frac{-3y}{-3y} + \frac{9yz}{-3y}\right)$$

$$= 2x(1-3z) - 3y(1-3z)$$

$$= (1-3z)\left(\frac{2x(1-3z)}{(1-3z)} - \frac{3y(1-3z)}{(1-3z)}\right)$$

$$= (1-3z)(2x-3y)$$

Example

Factorise the following by grouping:

$$ax - x^2 - a + x$$

If the third term has been negative, take the common factor with the negative sign.

$$ax - x^2 - a + x = x\left(\frac{ax}{x} - \frac{x^2}{x}\right) - 1\left(\frac{-a}{-1} + \frac{x}{-1}\right)$$

$$= x(a - x) - 1\,(a - x)$$

$$= (a - x)\left(\frac{x(a - x)}{(a - x)} - \frac{1(a - x)}{(a - x)}\right)$$

$$= (a - xb)(x - 1)$$

Difference of two squares

The "difference of two squares" is a special factorisation formula in algebra. It's used when you have two terms that are both perfect squares and are separated by a subtraction sign. The formula is:

Expansion →

$$(x-y)(x+y) = x^2 - y^2$$

← Factorisation

Note:
the sum of two squares does not factorise into two real linear factors.

For example:

$$x^2 - 9 = (x^2 - 3^2) = (x-3)(x+3)$$

$$4x^2 - 25y^2 = ((2x)^2 - (5y)^2) = (2x-5y)(2x+5y)$$

$$1 - x^2 = (1^2 - x^2) = (1-x)(1+x)$$

Example

Factorise the following expressions.

a) $25 - x^2$

b) $9x^2 - y^2$

c) $4x^2 - 81y^2$

d) $49x^2 - 1$

a) $25 - x^2 = (5 - x)(5 + x)$

b) $9x^2 - y^2 = (3x - y)(3x + y)$

c) $4x^2 - 81y^2 = (2x - 9y)(2x + 9y)$

d) $49x^2 - 1 = (7x - 1)(7x + 1)$

Example

Factorise the following expressions.

a) $3 - 27x^2$

b) $8x^2y + 2y^3$

In factoring, finding the common factor is always the first step.

a) $3 - 27x^2 = 3(\frac{3}{3} - \frac{27x^2}{3})$

$= 3(1 - 9x^2)$

$= 3(1 - 3x)(1 + 3x)$

b) $8x^2y + 2y^3 = 2y(\frac{8x^2y}{2y} + \frac{2y^3}{2y})$

$= 2y(4x^2 + y^2)$

$(4x^2 + y^2)$

Is not the difference between two squares

Therefore, the final answer is:

$= 2y(4x^2 + y^2)$

Factorise $ax^2 + bx + c$

To factorise a quadratic trinomial, we need to find two binomials whose product is equal to the quadratic trinomial. The general form of a quadratic trinomial is $ax^2 + bx + c$, where a, b and c are constants.

To factorise this expression, we need to find two binomials of the form $(px + q)$ and $(rx + s)$ such that their product is equal to the quadratic trinomial.

$$ax^2 + bx + c = (px + q)(rx + s)$$

The general steps to solve a quadratic by factorisation are as follows:

1. **If necessary, rearrange the quadratic into the form $ax^2 + bx + c$**
2. **Multiply a by c, then find 2 factors of ac whose sum is equal to b.**
3. **Once we have the correct factors, replace bx with these factors.**
4. **Take a common factor of the first two terms then do the same for the last two terms. You should notice that you have two brackets the same.**

Example

Factorise the following trinomial expression.

$$x^2 + 6 - 5x$$

Step 1

If necessary, rearrange the quadratic into the form $ax^2 + bx + c = 0$

$x^2 + 6 - 5x \quad \rightarrow \quad x^2 - 5x + 6$

Step 2

Multiply a by c, then find 2 factors of ac whose sum is equal to b.

$1 \times 6 = 6$

Factors of 6	**1,6**	**$-1. -6$**	**2, 3**	**$-2, -3$**
sum is equal to b.				**$-2 + (-3) = -5$**

Step 3

Once we have the correct factors, replace bx with these factors.

$x^2 - 2x - 3x + 6$

Step 4

Take a common factor of the first two terms then do the same for the last two terms. You should notice that you have two brackets the same.

$x^2 - 2x - 3x + 6 \quad \rightarrow \quad x(x-2) - 3(x-2) \quad \rightarrow \quad (x-2)(x-3)$

Example

Factorise the following trinomial expression.

$$2x^2 - 9x - 5$$

$$2x^2 - 9x - 5$$

$2 \times (-5) = -10$

Factors of -10	$1, -10$	-1, 10	-2, 5	2, -5
sum is equal to b.	$-10 + 1 = -9$			

$$2x^2 - 10x + x - 5$$

$$2x^2 - 10x + x - 5 \quad \rightarrow \quad 2x(x - 5) + (x - 5) \quad \rightarrow \quad (x - 5)(2x + 1)$$

Example

Factorise the following trinomial expression.

$$9x^2 + 4 - 12x$$

$$9x^2 + 4 - 12x \quad \longrightarrow \quad 9x^2 - 12x + 4$$

$9 \times 4 = 36$

Factors of 36	$1, 36$	$-1, -36$	2,18	-2,-18	3,12	-3,-12	4,9	-4,-9	6,6	-6,-6
sum is equal to b										$-6+(-6)=-12$

$$9x^2 - 12x + 4$$

$$9x^2 - 6x - 6x + 4 \quad \longrightarrow \quad 3x(3x-2) - 2(3x-2) \quad \longrightarrow \quad (3x-2)(3x-2)$$

$$\text{or} \rightarrow (3x-2)^2$$

Example

Factorise the following trinomial expression.

$$x^2 - 6 - 5x$$

$x^2 - 6 - 5x \quad \rightarrow \quad x^2 - 5x - 6$

$1 \times (-6) = -6$

Factors of -6	**-1× 6**	$\mathbf{1 \times (-6)}$	**-2** $\times$ **3**	$\mathbf{2 \times (-3)}$
sum is equal to b.		$1 + (-6) = -5$		$-$

$x^2 - 5x - 6$

$x^2 + x - 6x - 6 \quad \rightarrow \quad x(x+1) - 6(x+1) \quad \rightarrow \quad (x+1)(x-6)$

Example

Factorise the following trinomial expression.

$$2x^2 + 12 + 11x$$

$2x^2 + 12 + 11x \rightarrow 2x^2 + 11x + 12$

$2 \times 12 = 24$

Factors of 24	$\mathbf{1 \times 24}$	$\mathbf{2 \times 12}$	$\mathbf{3 \times 8}$
sum is equal to b.			$3 + 8 = 11$

$2x^2 + 11x + 12$

$2x^2 + 8x + 3x + 12 \rightarrow 2x(x + 4) + 3(x + 4) \rightarrow (x + 4)(2x + 3)$

Example

Factorise the following expressions.

$$4x^3 + 8x^2 - 12x$$

In factoring, finding the common factor is always the first step.

$$4x^3 + 8x^2 - 12x = 4x(x^2 + 2x - 3)$$

$x^2 + 2x - 3$

$1 \times (-3) = -3$

Factors of -3	$\mathbf{1 \times (-3)}$	$\mathbf{-1 \times 3}$
sum is equal to b.	$1 + 3 \neq 2$	$-1 + 3 = -2$

$x^2 - 2x - 3$

$x^2 - x + 3x - 3 \quad \rightarrow \quad x(x-1) + 3(x-1) \quad \rightarrow \quad (x-1)(x+3)$

Therefore:

$$4x^3 + 8x^2 - 12x = 4x(x-1)(x+3)$$

Example

Factorise the following expressions.

$$3x^3 + 6x^2 + 9x$$

$$3x^3 + 6x^2 + 9x \ = 3x(x^2 + 2x + 3)$$

$$x^2 + 2x + 3$$

$$1 \times 3 = 3$$

Factors of 3	$\mathbf{1 \times 3}$	$\mathbf{-1 \times (-3)}$
sum is equal to b.	$1 + 3 \neq 2$	$-1 + (-3) \neq 2$

The sum of any pair of factors of the number 3 is not equal to 2 (the value of B).
Therefore, the three sentences are not factorisable.

Therefore:

$$3x^3 + 6x^2 + 9x \ = 3x(x^2 + 2x + 3)$$

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Factorisation

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