Quadratic equations

A quadratic equation is an equation of the form $\boldsymbol{ax^2 + bx + c = 0}$ where a, b, and c are constants, a $\neq$0.

The roots or solutions of a quadratic equation are the values of x that satisfy the equation.

There are three methods to solve quadratic equations:

- Factorisation
- Completing the square
- Using the quadratic formula

Factorisation

The roots of a quadratic equation can be found by factorising the quadratic expression into two brackets and setting each bracket to 0 to derive two solutions, called roots

The general steps to solve a quadratic by factorisation are as follows:

1. **If necessary, rearrange the quadratic into the form $ax^2 + bx + c = 0$**
2. **Multiply a by c, then find 2 factors of ac whose sum is equal to b.**
3. **Once we have the correct factors, replace bx with these factors.**
4. **Take a common factor of the first two terms then do the same for the last two terms. You should notice that you have two brackets the same.**
5. **Use the Null Factor law: If $p \times q = 0$ then $p = 0$ or $q = 0$.**

Example

Solve the following equation by factorisation.

$$x^2 + 6 = 5x$$

Step 1 If necessary, rearrange the quadratic into the form $ax^2 + bx + c = 0$

$$x^2 + 6 = 5x \quad \longrightarrow \quad x^2 - 5x + 6 = 0$$

Step 2 Multiply a by c, then find 2 factors of ac whose sum is equal to b.

$$1 \times 6 = 6$$

Factors of 6	6, 1	-6,-1	2,3	-2,-3
sum is equal to b.				$-2 + (-3) = -5$

Step 3 Once we have the correct factors, replace bx with these factors.

$$x^2 - 2x - 3x + 6 = 0$$

Step 4 Take a common factor of the first two terms then do the same for the last two terms. You should notice that you have two brackets the same.

$$x^2 - 2x - 3x + 6 = 0 \longrightarrow x(x-2) - 3(x-2) = 0 \longrightarrow (x-2)(x-3) = 0$$

Step 5 Use the Null Factor law: If $p \times q = 0$ then $p = 0$ or $q = 0$.

$$(x-2)(x-3) = 0 \longrightarrow \begin{matrix} x - 2 = 0 & x = 2 \\ x - 3 = 0 & x = 3 \end{matrix}$$

The equation has two distinct real roots.

Example

Solve the following equation by factorisation.

$$2x^2 - 9x = 5$$

$2x^2 - 9x = 5 \quad \longrightarrow \quad 2x^2 - 9x - 5 = 0$

$2 \times (-5) = -10$

Factors of -10	$-10, 1$	10,-1	-2,5	2,-5
sum is equal to b.	$-10 + 1 = -9$			

$2x^2 - 9x - 5 = 0$

$2x^2 - 10x + x - 5 = 0 \longrightarrow 2x(x-5) + (x-5) = 0 \longrightarrow (x-5)(2x+1) = 0$

$(x-5)(2x+1) = 0 \longrightarrow$

$x - 5 = 0 \qquad x = 5$

$2x + 1 = 0 \qquad x = \dfrac{-1}{2}$

The equation has two distinct real roots.

Example

Solve the following equation by factorisation.

$$x^2 - \frac{2x}{3} - 7 = 0$$

$$x^2 - \frac{2x}{3} - 7 = 0 \xrightarrow{\times 3} 3x^2 - 2x - 21 = 0$$

$3 \times (-21) = -63$

Factors of -63	$-63, 1$	$63, -1$	-9, 7	9,-7
sum is equal to b.			$-9 + 7 = -2$	

$$3x^2 - 2x - 21 = 0$$

$$3x^2 - 9x + 7x - 21 = 0 \longrightarrow 3x(x-3) + 7(x-3) = 0 \longrightarrow (x-3)(3x+7) = 0$$

$$(x-3)(3x+7) = 0 \longrightarrow \begin{array}{ll} x - 3 = 0 & x = 3 \\ 3x + 7 = 0 & x = \frac{-7}{3} \end{array}$$

The equation has two distinct real roots.

Example

Solve the following equation by factorisation.

$$9x^2 + 4 = 12x$$

$9x^2 + 4 = 12x \quad \rightarrow \quad 9x^2 - 12x + 4 = 0$

$9 \times 4 = 36$

Factors of 36	36, 1	-36,-1	-6,-6	-4,-9
sum is equal to b.			$-6 + (-6) = -12$	

$9x^2 - 12x + 4 = 0$

$9x^2 - 6x - 6x + 4 = 0 \rightarrow 3x(3x - 2) - 2(3x - 2) = 0 \rightarrow (3x - 2)(3x - 2) = 0$

$(3x - 2)^2 = 0 \quad \rightarrow$

$3x - 2 = 0 \quad x = \frac{2}{3}$

$3x - 2 = 0 \quad x = \frac{2}{3}$

The equation has one real roots. (Repeated root)

Example

Solve the following equation by factorisation.

$$3x^2 = 12x$$

$3x^2 = 12x$ $\quad$ $3x^2 - 12x = 0$

$3x^2 - 12x = 0$ $\quad$ $3x(x-4) = 0$

→ $3x = 0$ $\quad$ $x = 0$

→ $(x-4) = 0$ $\quad$ $x = 4$

The equation has 2 distinct real roots.

Example

Solve the following equation by factorisation.

$$\frac{5}{2}x = 5x^2$$

$$\frac{5}{2}x - 5x^2 = 0 \quad \rightarrow \quad 5x(\frac{1}{2} - x) = 0$$

$$5x(\frac{1}{2} - x) = 0$$

$\rightarrow$ $5x = 0$ $x = 0$

$\rightarrow$ $\left(\frac{1}{2} - x\right) = 0$ $x = \frac{1}{2}$

The equation has 2 distinct real roots.

Completing the square

Another method for solving quadratic equations is completing the square which involves rearranging the quadratic expression into a completed square form and then solving for x by taking the square root of both sides.

The general steps to solve a quadratic by completing the square as follows:

1. **If necessary, rearrange the quadratic into the form $ax^2 + bx + c = 0$, then rewrite the quadratic expression as a complete square.**

$$a(x+p)^2 + q = 0 \quad \text{where } p = \frac{b}{2a} \text{ and } q = c - \frac{b^2}{4a}$$

2. **Solve the equation.** $a(x+p)^2 + q = 0$

$$a(x+p)^2 = -q \qquad (x+p)^2 = \frac{-q}{a} \qquad x + p = \pm\sqrt{\frac{-q}{a}} \qquad x = \pm\sqrt{\frac{-q}{a}} - p$$

Example

Solve the following equation by completing the square.

$$x^2 - 2x - 4 = 0$$

Step 1

If necessary, rearrange the quadratic into the form $ax^2 + bx + c = 0$, then rewrite the quadratic expression as a complete square.

$$a(x+p)^2 + q = 0 \text{ where } p = \frac{b}{2a} \text{ and } q = c - \frac{b^2}{4a}$$

$x^2 - 2x - 4 = 0$

$a = 1$

$b = -2$

$c = -4$

$$p = \frac{b}{2a} = \frac{(-2)}{2(1)} = -1$$

$$q = c - \frac{b^2}{4a} = -4 - \frac{(-2)^2}{4(1)} = -5$$

$$a(x+p)^2 + q = 0$$

$$1(x + (-1))^2 + (-5) = 0$$

Step 2

Making x the subject of the formula to solve the equation. $a(x-p)^2 + q = 0$

$$a(x+p)^2 = -q \rightarrow (x+p)^2 = \frac{-q}{a} \rightarrow (x+p) = \pm\sqrt{\frac{-q}{a}} \rightarrow x = \pm\sqrt{\frac{-q}{a}} - p$$

$$(x-1)^2 - 5 = 0 \rightarrow (x-1)^2 = 5 \rightarrow (x-1) = \pm\sqrt{5} \rightarrow x = \pm\sqrt{5} + 1$$

$x = \sqrt{5} + 1 = 3.24$ and $x = -\sqrt{5} + 1 = -1.24$

The equation has 2 distinct real roots.

Example

Solve the following equation by completing the square.

$$2x^2 + 16x = 7$$

$$2x^2 + 16x - 7 = 0$$

$a = 2$

$b = 16$

$c = -7$

$$\boldsymbol{p = \frac{b}{2a}}$$

$$= \frac{(16)}{2(2)} = 4$$

$$\boldsymbol{q = c - \frac{b^2}{4a}}$$

$$= -7 - \frac{(16)^2}{4(2)}$$

$$= -39$$

$$\boldsymbol{a(x+p)^2 + q = 0}$$

$$2(x+4)^2 + (-39) = 0$$

$$2(x+4)^2 - 39 = 0 \rightarrow (x+4)^2 = \frac{39}{2} \rightarrow (x+4) = \pm\sqrt{\frac{39}{2}} \rightarrow x = \pm\sqrt{\frac{39}{2}} - 4$$

$x = \sqrt{\frac{39}{2}} - 4 = 0.42$ and $x = -\sqrt{\frac{39}{2}} - 4 = -8.42$

The equation has 2 distinct real roots.

Example

Solve the following equation by completing the square.

$$4x^2 + \frac{1}{4} = 2x$$

$$4x^2 - 2x + \frac{1}{4} = 0$$

$a = 4$

$b = -2$

$c = +\frac{1}{4}$

$$\boldsymbol{p = \frac{b}{2a}}$$

$$= \frac{(-2)}{2(4)} = \frac{-1}{4}$$

$$\boldsymbol{q = c - \frac{b^2}{4a}}$$

$$= \frac{1}{4} - \frac{(-2)^2}{4(4)}$$

$$= 0$$

$$\boldsymbol{a(x+p)^2 + q = 0}$$

$$4(x + (\frac{-1}{4}))^2 + 0 = 0$$

$$4(x - (\frac{1}{4}))^2 + 0 = 0 \rightarrow \left(x - \frac{1}{4}\right)^2 = \frac{0}{4} \rightarrow (x - \frac{1}{4}) = \pm\sqrt{0} \rightarrow x = \pm 0 + \frac{1}{4}$$

$x = 0 + \frac{1}{4} = 0.25$ and $x = 0 + \frac{1}{4} = 0.25$

The equation has 1 real root.
(repeated root)

Example

Solve the following equation by completing the square.

$$2x^2 + 3x + 7 = 0$$

$2x^2 + 3x + 7 = 0$

$a = 2$

$b = 3$

$c = 7$

$$\boldsymbol{p = \frac{b}{2a}}$$

$$= \frac{(3)}{2(2)} = \frac{3}{4}$$

$$\boldsymbol{q = c - \frac{b^2}{4a}}$$

$$= 7 - \frac{(3)^2}{4(2)}$$

$$= \frac{47}{8}$$

$$\boldsymbol{a(x-p)^2 + q = 0}$$

$$2(x + \frac{3}{4})^2 + \frac{47}{8} = 0$$

$$2\left(x + \frac{3}{4}\right)^2 + \frac{47}{8} = 0 \rightarrow \left(x + \frac{3}{4}\right)^2 = \frac{-\frac{47}{8}}{2} \rightarrow (x + \frac{3}{4}) = \pm\sqrt{-\frac{47}{16}} \rightarrow$$

Negative numbers do not have square roots.

This equation has no roots.

Quadratic formula

The quadratic formula is used to solve quadratic equations of the form $ax^2 + bx + c = 0$ where a, b and c are constants. The formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The general steps to solve a quadratic by using the quadratic formula as follows:

1. **If necessary, rearrange the quadratic into the form $ax^2 + bx + c = 0$**
2. **Use the quadratic formula to find the roots of the equation.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Solve the following equation by using the quadratic formula

$$2x^2 - 9x = 5$$

Step 1

If necessary, rearrange the quadratic into the form $ax^2 + bx +$

$$2x^2 - 9x = 5 \qquad 2x^2 - 9x - 5 = 0$$

$a = 2$

$b = -9$

$c = -5$

Step 2

Use the quadratic formula to find the roots of the equation.

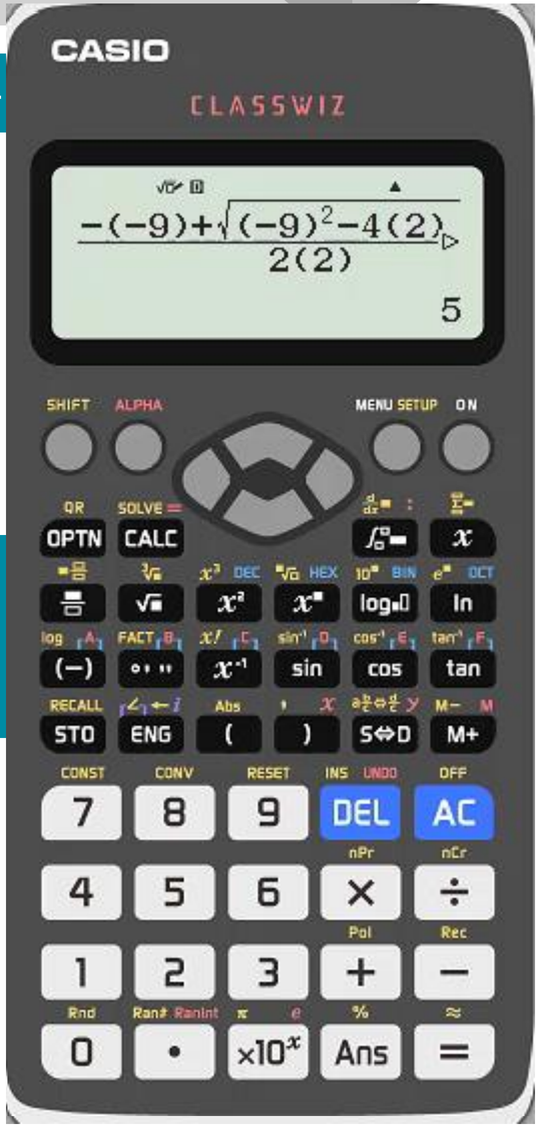
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(-5)}}{2(2)}$$

$x = 5$

$x = \frac{-1}{2} = -0.5$

Two distinct roots



Example Solve the following equation by using the quadratic formula

$$9x^2 + 1 = 6x$$

Step 1 **If necessary, rearrange the quadratic into the form $ax^2 + bx +$**

$$9x^2 + 1 = 6x \qquad 9x^2 - 6x + 1 = 0$$

$a = 9$

$b = -6$

$c = +1$

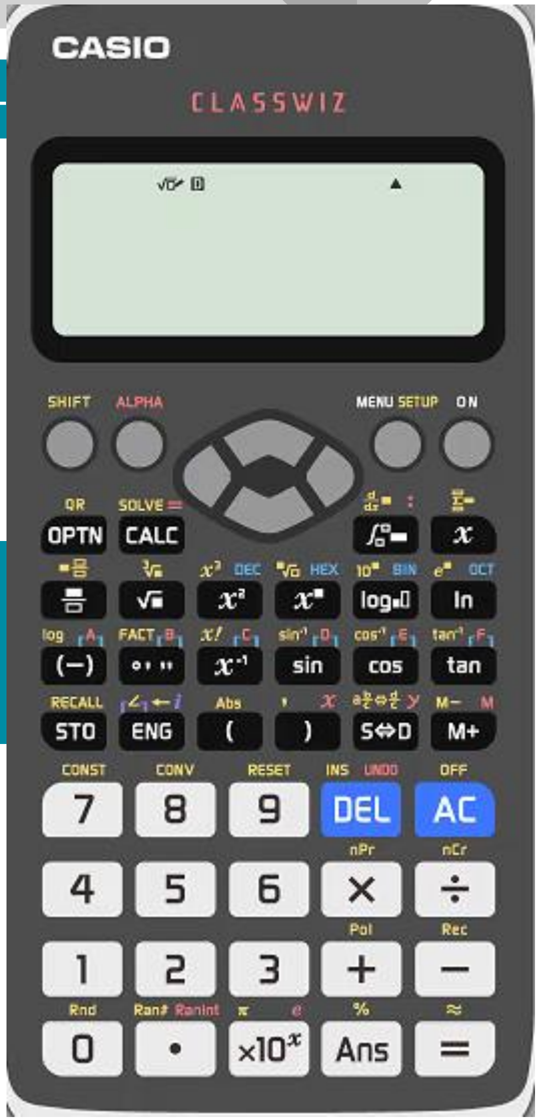
Step 2 **Use the quadratic formula to find the roots of the equation.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(1)}}{2(9)} \qquad x = \frac{1}{3}$$

$$x = \frac{6 \pm 0}{2(9)} \qquad x = \frac{1}{3}$$

One real root
(repeated root)



Example

Solve the following equation by using the quadratic formula

$$x - 2x^2 = 1$$

Step 1

If necessary, rearrange the quadratic into the form $ax^2 + bx +$

$$x - 2x^2 = 1 \qquad -2x^2 + x - 1 = 0$$

$$a = -2$$

$$b = 1$$

$$c = -1$$

Step 2

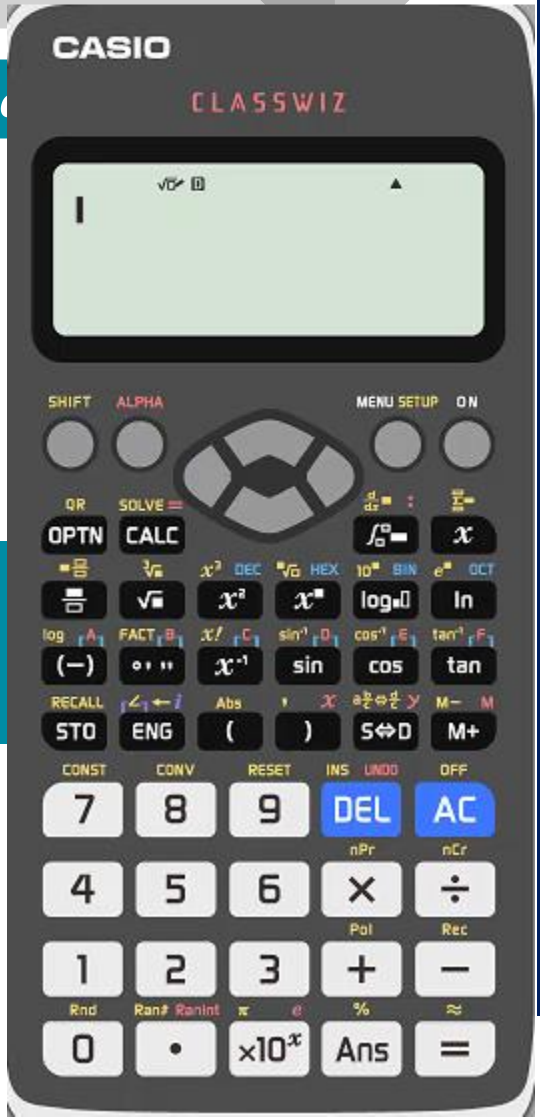
Use the quadratic formula to find the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(-2)(-1)}}{2(-2)}$$

$$x = \frac{-1 \pm \sqrt{-7}}{-4}$$

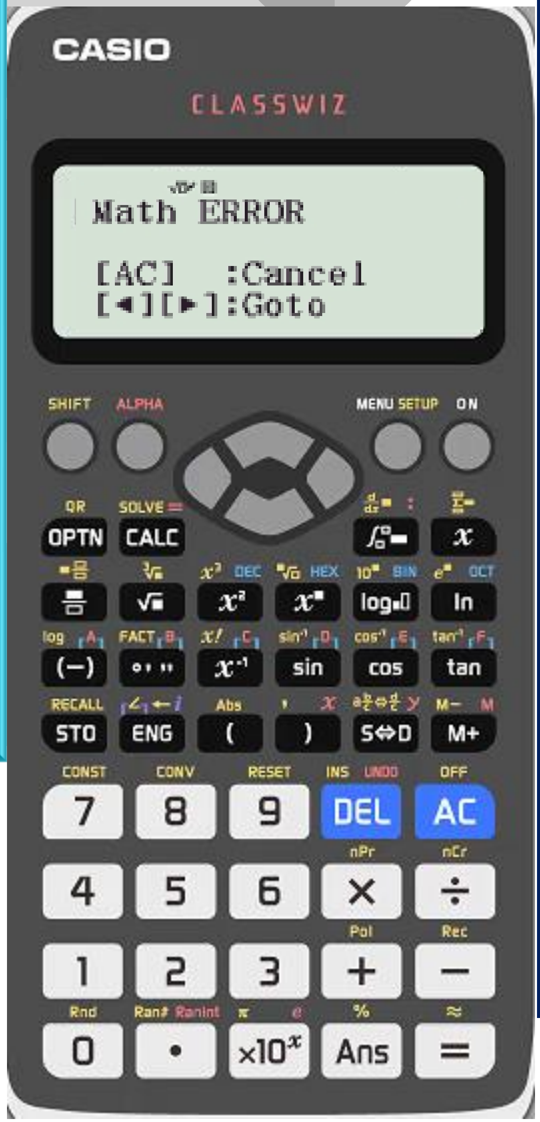
The square root of negative numbers is undefined.



When using calculators, you may encounter two types of errors:
syntax errors and **math errors**.

A **syntax error** occurs when the calculator is unable to interpret the input you have given it.
This can happen when you enter an expression that is not well-formed, such as forgetting to close a parenthesis or using an operator in the wrong order.

A **math error** occurs when the calculator is unable to perform a mathematical operation.
This can happen when you attempt to divide by zero, take the square root of a negative number, or perform some other operation that is undefined or outside the calculator's capabilities.



$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(-2)(-1)}}{2(-2)}$$

$$x = \frac{-1 \pm \sqrt{-7}}{-4}$$

The square root of negative numbers is undefined.

This equation has **no real roots**.

Example

Solve the following equation:

$$\frac{2}{x+2}+\frac{3}{2x-1}=1$$

$$\frac{2}{x+2}\times\frac{2x-1}{2x-1}+\frac{3}{2x-1}\times\frac{x+2}{x+2}=1$$

To adding fractions, need to same denominator (common denominator).

$$\frac{2(2x-1)}{(x+2)(2x-1)}+\frac{3(x+2)}{(2x-1)(x+2)}=1$$

$$\frac{2(2x-1)+3(x+2)}{(x+2)(2x-1)}=1$$

$$\frac{2(2x-1)+3(x+2)}{\cancel{(x+2)(2x-1)}}\times\cancel{(x+2)(2x-1)}=1\times(x+2)(2x-1)$$

To remove the denominator, multiply both sides by the denominator.

$$4x-2+3x+6=2x^2-x+4x-2$$

$$7x+4=2x^2+3x-2$$

$$0=2x^2+3x-2-7x-4$$

$$2x^2-4x-6=0 \qquad x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-6)}}{2(2)}$$

$x=-1$

$x=3$

Example

Solve the following equation:

$$\frac{x}{x+2} = \frac{3}{x-6}$$

$$\frac{x}{x+2} \times (x+2) \times (x-6) = \frac{3}{x-6} \times (x+2) \times (x-6)$$

To remove the denominators, multiply both sides by the denominators.

$$x^2 + 2x = 3x + 6$$

$$x^2 + 2x - 3x - 6 = 0$$

$$x^2 - x - 6 = 0$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)a}$$

$$x = 3$$

$$x = -2$$

mathematicschapters.com

Solve quadratic equations

by factorisation, completing the square and by use of the quadratic formula

mathematicschapters.com