Surd

A surd is an irrational number that can't be simplified to remove a square root (or cube root, etc.).
Surds typically involve roots of numbers that do not result in a whole number or a fraction when simplified.

Simplifying surds means expressing a surd in its simplest form, where no prim number remains under the root symbol with power greater than or equal to the root.
To simplify a surd, you break down the number under the root into its prime factors.

Note:

$$\sqrt[n]{a^n} = a$$

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Example

Simplify the following expressions:

$a)\ \sqrt{9}$

$b)\ \sqrt{27}$

$c)\ \sqrt{45}$

$d)\ \sqrt{81}$

$\boldsymbol{a})\ \sqrt{\mathbf{9}} = \sqrt{3^2}$

$= 3$

$\boldsymbol{c})\ \sqrt{\mathbf{45}} = \sqrt{3^2 \times 5}$

$= \sqrt{3^2} \times \sqrt{5}$

$= 3 \times \sqrt{5}$

$\boldsymbol{b})\ \sqrt{\mathbf{27}} = \sqrt{3^3}$

$= \sqrt{3^2 \times 3}$

$= \sqrt{3^2} \times \sqrt{3}$

$= 3 \times \sqrt{3}$

$\boldsymbol{d})\ \sqrt{\mathbf{81}} = \sqrt{3^4}$

$= \sqrt{3^2 \times 3^2}$

$= \sqrt{3^2} \times \sqrt{3^2}$

$= 3 \times 3 = 9$

Example

Simplify the following expressions:

$a)\ \sqrt{25}$ $\quad c)\ \sqrt{50}$

$b)\ \sqrt{125}$ $\quad d)\ \sqrt{625}$

$$\boldsymbol{a)}\ \sqrt{\mathbf{25}} = \sqrt{5^2} = 5$$

$$\boldsymbol{b)}\ \sqrt{\mathbf{125}} = \sqrt{5^3} = \sqrt{5^2 \times 5} = \sqrt{5^2} \times \sqrt{5} = 5 \times \sqrt{5}$$

$$\boldsymbol{c)}\ \sqrt{\mathbf{50}} = \sqrt{2 \times 5^2} = \sqrt{2} \times \sqrt{5^2} = \sqrt{2} \times 5$$

$$\boldsymbol{d)}\ \sqrt{\mathbf{625}} = \sqrt{5^4} = \sqrt{5^2 \times 5^2} = \sqrt{5^2} \times \sqrt{5^2} = 5 \times 5 = 25$$

Example

Simplify the following expressions:

$a)\ \sqrt{0.04}$ $\qquad$ $c)\ \sqrt{200}$

$b)\ \sqrt{0.0008}$ $\qquad$ $d)\ \sqrt{16000}$

Note: $100 = 10^2\ , \quad 1000 = 10^3\ , \quad 10000 = 10^4\ , \ldots$

$$\boldsymbol{a)}\ \sqrt{\mathbf{0.04}} = \sqrt{\frac{4}{100}} = \sqrt{\frac{2^2}{10^2}} = \frac{\sqrt{2^2}}{\sqrt{10^2}} = \frac{2}{10} = 0.2$$

$$\boldsymbol{b)}\ \sqrt{\mathbf{0.0008}} = \sqrt{\frac{8}{10000}} = \sqrt{\frac{2^3}{10^4}} = \frac{\sqrt{2^2}\times\sqrt{2}}{\sqrt{10^2}\times\sqrt{10^2}} = \frac{2\sqrt{2}}{10\times 10} = \frac{\sqrt{2}}{50}$$

$$\boldsymbol{c)}\ \sqrt{\mathbf{200}} = \sqrt{2\times 10^2} = \sqrt{2}\times\sqrt{10^2} = \sqrt{2}\times 10$$

$$\boldsymbol{d)}\ \sqrt{\mathbf{16000}} = \sqrt{2^4\times 10^3} = \sqrt{2^2}\times\sqrt{2^2}\times\sqrt{10^2}\times\sqrt{10} = 2\times 2\times 10\times\sqrt{10} = 40\times\sqrt{10}$$

Operations with Surds

Like and Unlike Surds

Two surds are "like" if the parts inside the roots are identical and have the same symbol root.

For example:

- $\sqrt{2}$, $3\sqrt{2}$, $\frac{\sqrt{2}}{5}$, $-7\sqrt{2}$ are like surds.
- $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt{5}$, $2\sqrt{3}$ are not like surds.

Addition and Subtraction of Surds

Only like surds can be added or subtracted.

$$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$$
$$a\sqrt{b} - c\sqrt{b} = (a - c)\sqrt{b}$$

Note: Always simplify surds before adding or subtracting them.

Multiplication and division of surds

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$$

$$a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}}$$

Operations with Surds

Note:

$$(\sqrt{a})^2 = \sqrt{a^2} = a$$

$$\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = \sqrt{a^2} = a$$

Therefore:

$$(\sqrt{a})^2 = \text{a}$$

$$\sqrt{a} \times \sqrt{a} = a$$

Example

Do not use a calculator in this question.

Simplify $3\sqrt{2} - 5\sqrt{3} + 2\sqrt{2} - \sqrt{3}$.

$\mathbf{3\sqrt{2} - 5\sqrt{3} + 2\sqrt{2} - \sqrt{3}} = 5\sqrt{2} - 6\sqrt{3}$

Example

Do not use a calculator in this question.

Simplify $3\sqrt{28} - 7\sqrt{7}$.

Note: Always simplify surds before adding or subtracting them.

$$\mathbf{3\sqrt{28} - 7\sqrt{7}} = 3\sqrt{2^2 \times 7} - 7\sqrt{7}$$

$$= 3\sqrt{2^2} \times \sqrt{7} - 7\sqrt{7}$$

$$= 3(2) \times \sqrt{7} - 7\sqrt{7}$$

$$= 6\sqrt{7} - 7\sqrt{7}$$

$$= -\sqrt{7}$$

Example

Do not use a calculator in this question.

Expand and simplify $(3-5\sqrt{7})^2$.

Perfect Square Formula $(a+b)^2 = (a)^2 + 2(a)(b) + (b)^2$

$$\mathbf{(3-5\sqrt{7})^2} = (3)^2 + 2(3)\left(-5\sqrt{7}\right) + (5\sqrt{7})^2$$

$$= 9 - 30\sqrt{7} + (5)^2\sqrt{7^2}$$

$$= 9 - 30\sqrt{7} + 25 \times 7$$

$$= 184 - 30\sqrt{7}$$

Example

Do not use a calculator in this question.

Expand and simplify $(2-3\sqrt{2})(2+3\sqrt{2})$.

Difference between two squares formula $(a-b)(a+b)=a^2-b^2$

$$\mathbf{(2-3\sqrt{2})(2+3\sqrt{2})} = (2)^2-(3\sqrt{2})^2$$

$$= 4-(3)^2\times\sqrt{2^2}$$

$$= 4-9\times 2$$

$$= -14$$

Example

Do not use a calculator in this question.

Simplify $\frac{8\sqrt{18}}{2\sqrt{2}}$.

$$\frac{\mathbf{8\sqrt{18}}}{\mathbf{2\sqrt{2}}} = \frac{8}{2} \times \sqrt{\frac{18}{2}}$$

$$= 4\sqrt{9}$$

$$= 4 \times 3 = 12$$

Rationalise the denominator

1. Rationalising when there is a single surd in the denominator:

If the denominator contains a simple surd, you can rationalise the denominator by multiplying both the numerator and denominator by the same surd.

$$\frac{1}{\sqrt{c}} \times \frac{\sqrt{c}}{\sqrt{c}} = \frac{\sqrt{c}}{c}$$

For example: Rationalise the denominators of:

$$\frac{\mathbf{2}}{\sqrt{\mathbf{3}}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2\sqrt{3}}{3}$$

$$\frac{\mathbf{1}}{\mathbf{5}\sqrt{\mathbf{2}}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{5 \times (\sqrt{2})^2} = \frac{\sqrt{2}}{5 \times 2} = \frac{\sqrt{2}}{10}$$

Rationalise the denominator

2. Rationalizing when there is a binomial surd in the denominator:

If the fraction contains a sum of two terms in the denominator, we multiply both the top and bottom by the conjugate of the denominator.

The conjugate is where we change the sign in the middle of two terms.

$$\frac{1}{a-\sqrt{b}} \times \frac{a+\sqrt{b}}{a+\sqrt{b}} = \frac{a+\sqrt{b}}{(a)^2-(\sqrt{b})^2} = \frac{a+\sqrt{b}}{a^2-b}$$

Use the difference between two squares formula to simplify the denominator
$(a-b)(a+b) = a^2 - b^2$

For example: $\frac{\mathbf{2}}{\mathbf{3}-\sqrt{\mathbf{5}}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{2(3+\sqrt{5})}{(3)^2-(\sqrt{5})^2} = \frac{2(3+\sqrt{5})}{3^2-5} = \frac{2(3+\sqrt{5})}{4} = \frac{(3+\sqrt{5})}{2}$

$$\frac{\mathbf{1}}{\mathbf{3}\sqrt{\mathbf{2}}+\mathbf{1}} \times \frac{3\sqrt{2}-1}{3\sqrt{2}-1} = \frac{3\sqrt{2}-1}{(3\sqrt{2})^2-(1)^2} = \frac{3\sqrt{2}-1}{3^2(2)-1} = \frac{3\sqrt{2}-1}{18-1} = \frac{3\sqrt{2}-1}{17}$$

$$\frac{\mathbf{7}}{\sqrt{\mathbf{2}}+\mathbf{3}\sqrt{\mathbf{5}}} \times \frac{\sqrt{2}-3\sqrt{5}}{\sqrt{2}-3\sqrt{5}} = \frac{7(\sqrt{2}-3\sqrt{5})}{(\sqrt{2})^2-(3\sqrt{5})^2} = \frac{7(\sqrt{2}-3\sqrt{5})}{2-(3^2)5} = \frac{7(\sqrt{2}-3\sqrt{5})}{2-45} = \frac{7\sqrt{2}-21\sqrt{5}}{-43}$$

Example

Rationalise the denominators of:

a) $\dfrac{7}{\sqrt{7}}$ b) $\dfrac{1}{5\sqrt{5}}$ c) $\dfrac{1-\sqrt{2}}{\sqrt{2}}$

$$\frac{1}{\sqrt{c}} \times \frac{\sqrt{c}}{\sqrt{c}} = \frac{\sqrt{c}}{c}$$

a) $\dfrac{\mathbf{7}}{\sqrt{\mathbf{7}}} \times \dfrac{\sqrt{7}}{\sqrt{7}}$

$$= \frac{7\sqrt{7}}{(\sqrt{7})^2}$$

$$= \frac{7\sqrt{7}}{7}$$

$$= \sqrt{7}$$

b) $\dfrac{\mathbf{1}}{\mathbf{5}\sqrt{\mathbf{5}}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$

$$= \frac{\sqrt{5}}{5(\sqrt{5})^2}$$

$$= \frac{\sqrt{5}}{5 \times 5}$$

$$= \frac{\sqrt{5}}{25}$$

c) $\dfrac{\mathbf{1}-\sqrt{\mathbf{2}}}{\sqrt{\mathbf{2}}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{(1-\sqrt{2})\sqrt{2}}{(\sqrt{2})^2}$$

$$= \frac{\sqrt{2}-(\sqrt{2})^2}{2}$$

$$= \frac{\sqrt{2}-2}{2}$$

Example

Rationalise the denominators of:

$a)\ \frac{7}{1+3\sqrt{5}}$ $b)\ \frac{5}{5\sqrt{5}-2}$ $c)\ \frac{1}{2-3\sqrt{2}}$

$$\frac{1}{a-\sqrt{b}} \times \frac{a+\sqrt{b}}{a+\sqrt{b}} = \frac{a+\sqrt{b}}{(a)^2-(\sqrt{b})^2} = \frac{a+\sqrt{b}}{a^2-b}$$

Use the difference between two squares formula to simplify the denominator

$$(a-b)(a+b) = a^2-b^2$$

$$\boldsymbol{a)}\ \frac{\mathbf{7}}{\mathbf{1+3\sqrt{5}}} \times \frac{1-3\sqrt{5}}{1-3\sqrt{5}}$$

$$= \frac{7(1-3\sqrt{5})}{(1)^2-(3\sqrt{5})^2}$$

$$= \frac{7(1-3\sqrt{5})}{1-3^2(5)}$$

$$= \frac{7-21\sqrt{5}}{-44}$$

$$\boldsymbol{b)}\ \frac{\mathbf{5}}{\mathbf{5\sqrt{5}-2}} \times \frac{5\sqrt{5}+2}{5\sqrt{5}+2}$$

$$= \frac{5(5\sqrt{5}+2)}{(5\sqrt{5})^2-(2)^2}$$

$$= \frac{5(5\sqrt{5}+2)}{5^2(5)-4}$$

$$= \frac{25\sqrt{5}+10)}{121}$$

$$\boldsymbol{c)}\ \frac{\mathbf{1}}{\mathbf{2-3\sqrt{2}}} \times \frac{2+3\sqrt{2}}{2+3\sqrt{2}}$$

$$= \frac{2+3\sqrt{2}}{(2)^2-(3\sqrt{2})^2}$$

$$= \frac{2+3\sqrt{2}}{4-3^2(2)}$$

$$= \frac{2+3\sqrt{2}}{-14}$$

Example

Do not use a calculator in this question.

Expand and simplify $(\frac{2+\sqrt{3}}{1-\sqrt{2}})^2$, giving your answer with a rational denominator.

Perfect Square Formula $(a+b)^2 = (a)^2 + 2(a)(b) + (b)^2$

$$(\frac{2+\sqrt{3}}{1-\sqrt{2}})^2 = \frac{(2+\sqrt{3})^2}{(1-\sqrt{2})^2}$$

$$= \frac{(2)^2+2(2)(\sqrt{3})+(\sqrt{3})^2}{(1)^2+2(1)(-\sqrt{2})+(-\sqrt{2})^2}$$

$$= \frac{4+4\sqrt{3}+3}{1-2\sqrt{2}+2}$$

$$= \frac{7+4\sqrt{3}}{3-2\sqrt{2}}$$

$$\frac{7+4\sqrt{3}}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}$$

$$= \frac{21+14\sqrt{2}+12\sqrt{3}+8\sqrt{6}}{(3)^2-(2\sqrt{2})^2}$$

$$= \frac{21+14\sqrt{2}+12\sqrt{3}+8\sqrt{6}}{9-(2)^2(2)}$$

$$= \frac{21+14\sqrt{2}+12\sqrt{3}+8\sqrt{6}}{1}$$

$$= 21+14\sqrt{2}+12\sqrt{3}+8\sqrt{6}$$